

External gates and transport in biased bilayer graphene

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(Dated: March 3, 2009)

We formulate a theory of transport in graphene bilayers in the weak momentum scattering regime in such a way as to take into account contributions to the electrical conductivity to leading and next-to-leading order in the scattering potential. The response of bilayers to an electric field cannot be regarded as a sum of terms due to individual layers. Rather, interlayer tunneling and coherence between positive- and negative-energy states give the main contributions to the conductivity. At low energies, the dominant effect of scattering on transport comes from scattering within each energy band, yet a simple picture encapsulating the role of collisions in a set of scattering times is not applicable. Coherence between positive- and negative-energy states gives, as in monolayers, a term in the conductivity which depends on the order of limits. The application of an external gate, which introduces a gap between positive- and negative-energy states, does not affect transport. Nevertheless the solution to the kinetic equation in the presence of such a gate is very revealing for transport in both bilayers and monolayers.

I. INTRODUCTION

The manufacture of carbon monolayers in the laboratory has generated unprecedented excitement. The unusual and sometimes baffling properties of graphene have exposed the community to novel science and supplied ideas for technological innovation.¹ This achievement was swiftly followed by the reliable manufacture of graphene bilayers and multilayers, which have become independent research areas in their own right. The electronic properties of bilayer graphene² have been the subject of recent reviews^{3,4} but the topic has been studied much less than single graphene layers. Large mobilities recently observed⁵ are among many factors justifying experimental and theoretical interest in bilayers. Also notable in these systems is a Berry phase of 2π accompanying an unconventional quantum Hall effect⁶ and predictions of Andreev reflection⁷ and superfluidity⁸ among other effects.^{9,10,11,12,13,14,15,16,17} Studies of graphene bilayers have focused on transport,^{18,19,20,21,22,23,24,25,26,27} compressibility,²⁸ impurities,^{29,30,31} electron-electron interactions,^{32,33,34,35} and band and electronic structure.^{36,37,38,39,40,41}

Graphene bilayers are interesting from a physics point of view because they are not merely the sum of two layers. Tunneling between layers is characterized by the large parameter t_{\perp} which is comparable to the Fermi energy at carrier densities n well beyond 10^{12} cm^{-2} . The energy spectrum of bilayers consists of four bands, two with positive energy and two with negative energy. The lower positive-energy band and the higher negative-energy band touch at $k = 0$. Although at low densities only one of these latter two bands is occupied, depending on whether the sample is doped with electrons or holes, a pseudospin cannot be defined in the same manner as in single graphene layers. This is due to the presence of the large tunneling parameter, which indicates a non-trivial interplay of carriers from both layers, particularly

important in steady-state processes. Furthermore, a gate potential opens a gap in the energy spectrum,³⁸ and independent control of this gap and of the carrier density can be achieved by using separate top and back gates.⁴²

This article presents a thorough investigation of transport in graphene bilayers, constructing a compact and straightforward framework for analyzing the structure of the steady-state density matrix in an electric field. The formalism takes the density operator and quantum Liouville equation as its starting point, treating all terms in the Hamiltonian on the same footing and using realistic scattering potentials. Calculation of the electrical current reveals the complex physics underlying bilayer transport. The conductivity contains a term which is a function of the carrier density n and is inversely proportional to the impurity density n_i , similar to the usual conductivity of metals and semiconductors. This term also depends on t_{\perp} , which can be thought of as indicating coherence between layers, while its dependence on n is different for long-range and short range impurities and it is in general not $\propto n$. The concept of a characteristic momentum scattering time is useful as an order of magnitude, but it is not possible to assign scattering times to carriers in different bands. Furthermore, an electric field couples positive and negative energy states, resulting in a term in the conductivity similar to that in single layers but of different magnitude because of the different winding number associated with energy dispersion in bilayers. It must also be borne in mind that positive- and negative-energy states involve carriers from both layers. In graphene monolayers the conductivity due to the coupling of positive- and negative-energy states is renormalized by scattering to twice its original value. In bilayers the effect of scattering on this term is considerably smaller than in monolayers. This term also depends on the order of limits, and work to date has not been able to extract its definite value. Fortunately, we will show that biased bilayer graphene ought to provide an answer. The external gate makes a contribution to the conduc-

tivity through an off-diagonal (Hall) term which would appear to exist without a magnetic field. This term also depends on the order of limits, but it is the sole contribution to the off-diagonal conductivity. Since crystal symmetry and Onsager relations imply that an off-diagonal conductivity cannot exist without a magnetic field this must indicate the correct and unambiguous order of limits. If any doubts persist experiment can surely resolve this issue, given that there is only one potential contribution to the off-diagonal conductivity. We note that Boltzmann transport theory has been formulated thoroughly for unbiased bilayer graphene.¹⁸ In this work we wish to consider both the ordinary (“Boltzmann”) contribution to the conductivity $\propto n/n_i$ and the contribution independent of n and n_i on the same footing. We also aim to determine unambiguously the role of the gate and the lessons to be learned from it.

We focus on extrinsic graphene in the weak momentum scattering regime, $\varepsilon_F \gg \hbar/\tau$, where ε_F is the Fermi energy and $\tau \propto n_i^{-1}$ is a characteristic momentum scattering time (for strong momentum scattering $\varepsilon_F \ll \hbar/\tau$.) Extrinsic graphene refers to the case $n > 0$ and may be in either the weak or the strong momentum scattering regime. Intrinsic graphene refers to the case $n = 0$ where the Fermi energy lies at the point where the bands touch, and is by definition in the strong momentum scattering regime. Enormous strides in sample quality make transport in the weak momentum scattering regime a timely undertaking.^{1,3,4} We assume low temperatures, where scattering due to charged impurities is important and may dominate, and where electron-electron scattering plays a secondary role. In the regime of weak momentum scattering studied here quantum interference effects are also expected to be negligible.⁴³ We stress once more that a conductivity independent of n and n_i was measured experimentally by taking $n \rightarrow 0$. The value obtained is characteristic of the *strong* momentum scattering regime and is referred to as the *minimum conductivity*.⁵ At the same time, theoretical research on *clean* samples finds an additional conductivity independent of n and n_i . It is this latter term, rather than the minimum conductivity, that is discussed in our work.

The outline of this article is as follows. In Section II we construct the kinetic equation for bilayer graphene taking the quantum Liouville equation as our starting point. We discuss in detail the role of a general elastic impurity potential in this kinetic equation specific to bilayers. In Section III we apply this equation to the study of transport in unbiased bilayers and identify contributions to the conductivity $\propto n/n_i$ and independent of n and n_i . In Section IV we discuss the role of the gate and show that, if treated in a naive manner, it can appear to yield an off-diagonal conductivity in the absence of a magnetic field. We discuss the implications of this finding for transport in graphene bilayers and monolayers.

II. KINETIC EQUATION

The formalism parallels that used in Ref. 44 and its exposition below is correspondingly abbreviated. The system is described by a density operator $\hat{\rho}$. Evaluation of $\hat{\rho}$ in the steady state allows one to calculate expectation values such as that of the velocity operator. Very generally, $\hat{\rho}$ obeys the quantum Liouville equation

$$\frac{d\hat{\rho}}{dt} + \frac{i}{\hbar} [\hat{H} + \hat{H}^E + \hat{U}, \hat{\rho}] = 0. \quad (1)$$

Here \hat{H} is the band Hamiltonian, $\hat{H}^E = e\mathbf{E} \cdot \hat{\mathbf{r}}$ the interaction with the external electric field \mathbf{E} , and \hat{U} the impurity potential. We project the Liouville equation onto a set of time-independent states of definite wave vector $\{|\mathbf{k}s\rangle\}$, in which $\langle \mathbf{k}s | \hat{\rho} | \mathbf{k}'s' \rangle \equiv \rho_{\mathbf{k}\mathbf{k}'}^{ss'} \equiv \rho_{\mathbf{k}\mathbf{k}'}$ and similarly $H_{\mathbf{k}\mathbf{k}'}$, $H_{\mathbf{k}\mathbf{k}'}^E$, and $U_{\mathbf{k}\mathbf{k}'}$. For bilayers the index s runs from 1 to 4 as will be shown below. We refer to $\rho_{\mathbf{k}\mathbf{k}'}$ as the density matrix. Matrix elements of $H_{\mathbf{k}\mathbf{k}'} = H_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'}$ are diagonal in \mathbf{k} but off-diagonal in ss' , and similar for $H_{\mathbf{k}}^E$. Matrix elements of $U_{\mathbf{k}\mathbf{k}'}$ are off-diagonal in \mathbf{k} . Elastic scattering is assumed and the average of terms $U_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}'\mathbf{k}}$ in the disorder potential over impurity configurations is $n_i |\bar{U}_{\mathbf{k}\mathbf{k}'}|^2 / V$, where V is the crystal volume and $\bar{U}_{\mathbf{k}\mathbf{k}'}$ the matrix element of the potential of a single impurity. $\rho_{\mathbf{k}\mathbf{k}'}$ has a part $f_{\mathbf{k}}$ diagonal in \mathbf{k} , and a part off-diagonal in \mathbf{k} . We will be interested in $f_{\mathbf{k}}$ since most operators related with steady-state processes are diagonal in \mathbf{k} . From the Liouville equation an effective equation is derived for $f_{\mathbf{k}}$ in the first Born approximation, valid for $\varepsilon_F \tau / \hbar \gg 1$

$$\frac{\partial f_{\mathbf{k}}}{\partial t} + \frac{i}{\hbar} [H_{\mathbf{k}}, f_{\mathbf{k}}] + \hat{J}(f_{\mathbf{k}}) = \Sigma_{\mathbf{k}}, \quad (2)$$

where the source term $\Sigma_{\mathbf{k}} = (e\mathbf{E}/\hbar) \cdot (\partial f_0 / \partial \mathbf{k})$, the equilibrium density matrix $f_0(H_{\mathbf{k}})$ is given by the Fermi-Dirac function and the scattering term takes the form

$$\hat{J}(f_{\mathbf{k}}) = (n_i / \hbar^2) \int_0^\infty dt' [\hat{U}, e^{-i\hat{H}t'/\hbar} [\hat{U}, \hat{f}] e^{i\hat{H}t'/\hbar}]_{\mathbf{k}\mathbf{k}}. \quad (3)$$

The low-energy bilayer graphene Hamiltonian is³

$$H_{\mathbf{k}} = \begin{pmatrix} 0 & \hbar v k e^{i\phi} & t_{\perp} & 0 \\ \hbar v k e^{-i\phi} & 0 & 0 & 0 \\ t_{\perp} & 0 & 0 & \hbar v k e^{-i\phi} \\ 0 & 0 & \hbar v k e^{i\phi} & 0 \end{pmatrix}. \quad (4)$$

Along the diagonal are two 2×2 submatrices which represent the Hamiltonians of individual layers, in which $v \approx 1.1 \times 10^6 \text{ m}^{-1}$ stands for the (constant) Fermi velocity of graphene. These layers are coupled by the interlayer tunneling parameter $t_{\perp} \approx 0.3 \text{ eV}$. Although Eq. (4) does not include the so-called *trigonal warping* terms, this model captures most of the important physics.³ If we define $\lambda_k = \sqrt{t_{\perp}^2 + 4\hbar^2 v^2 k^2}$, the energy eigenvalues can be labeled as $\varepsilon_{k1} = \frac{1}{2}(\lambda_k + t_{\perp}) = -\varepsilon_{k4}$ and $\varepsilon_{k2} = \frac{1}{2}(\lambda_k - t_{\perp}) = -\varepsilon_{k3}$, independent of the direction

of \mathbf{k} . The energies ε_{k1} and ε_{k2} are positive, whereas ε_{k3} and ε_{k4} are negative. The two bands ε_{k2} and ε_{k3} touch at $k = 0$. We work henceforth in the basis of eigenstates of $H_{\mathbf{k}}$ with the eigenvectors labeled $|u_{\mathbf{k}s}\rangle$ so that $|\mathbf{k}s\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}|u_{\mathbf{k}s}\rangle$. In this basis the Hamiltonian is diagonal and has the form $H_{\mathbf{k}} = \text{diag}(\varepsilon_{k1}, \varepsilon_{k2}, \varepsilon_{k3}, \varepsilon_{k4})$. Nevertheless, one must be careful in writing down the kinetic equation in this basis. As the basis functions, namely the eigenvectors of $H_{\mathbf{k}}$, depend on the magnitude and direction of the wave vector \mathbf{k} , the ordinary derivative with respect to \mathbf{k} must be replaced by the covariant derivative. The action of this covariant derivative for example on f_0 is given by $Df_0/D\mathbf{k} = \partial f_0/\partial\mathbf{k} - i[\mathcal{R}, f_0]$. The connection matrix \mathcal{R} which enters the covariant derivative has elements $\mathcal{R}_{ss'} = \langle u_{\mathbf{k}s} | i\partial u_{\mathbf{k}s'}/\partial\mathbf{k} \rangle$. It is easiest to work out the derivatives with respect to the magnitude k and polar angle θ of the wave vector,

$$\mathcal{R}^k = \frac{i\hbar v t_{\perp}}{\lambda_k^2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (5a)$$

$$\mathcal{R}^{\theta} = \frac{1}{k} \begin{pmatrix} 1 & -\frac{\hbar v k}{\lambda_k} & 0 & 0 \\ -\frac{\hbar v k}{\lambda_k} & 1 & -1 & 0 \\ 0 & -1 & 1 & -\frac{\hbar v k}{\lambda_k} \\ 0 & 0 & -\frac{\hbar v k}{\lambda_k} & 1 \end{pmatrix} \quad (5b)$$

These expressions will be needed when constructing the velocity operator, as well as when determining the source term in the kinetic equation in an electric field. Lastly, since the Hamiltonian of graphene bilayers and single layers does not depend on the spin of the particles, final results will contain a factor of 2 from the sum over the spin, as well as an additional factor of 2 which takes into account the twofold valley degeneracy of graphene. Therefore the final expressions for the conductivity must be multiplied by an overall factor of 4.

The matrix elements of the scattering potential \hat{U} due to a single impurity in the basis of $H_{\mathbf{k}}$ eigenstates are

$$\langle \mathbf{k}s | \hat{U} | \mathbf{k}'s' \rangle = \mathcal{U}_{\mathbf{k}\mathbf{k}'} M_{\mathbf{k}\mathbf{k}'}^{ss'} \equiv \mathcal{U}_{\mathbf{k}\mathbf{k}'} M_{\mathbf{k}\mathbf{k}'}, \quad (6)$$

where $\mathcal{U}_{\mathbf{k}\mathbf{k}'}$ are the matrix elements of \hat{U} between plane wave states. The band indices s and s' will be henceforth suppressed and quantities such as $M_{\mathbf{k}\mathbf{k}'}$ will be treated as matrices in the subspace spanned by the four bands under consideration (with the band index s the same as that introduced above.) The scattering term $\hat{J}(f_{\mathbf{k}})$ appearing in the kinetic equation simplifies considerably if we assume that the tunneling parameter $t_{\perp} \gg \hbar v k$. This assumption is valid for carrier densities up to approximately 10^{12} cm^{-2} . At these densities only one of the bands is occupied: for electron (hole) doping this is the band labeled 2 (3). We expand $M_{\mathbf{k}\mathbf{k}'}$ in the ratio $\hbar v k/t_{\perp}$ up to order 1, and we find that the term of order 1 vanishes identically. We label the incident wave vector by \mathbf{k} , the outgoing wave vector by \mathbf{k}' and the polar angle of

the outgoing wave vector \mathbf{k}' by θ' . If we define $\gamma = \theta' - \theta$ as the relative angle between incident and outgoing wave vectors, $M_{\mathbf{k}\mathbf{k}'}$ has the simple diagonal form

$$M_{\mathbf{k}\mathbf{k}'} = e^{-i\gamma} \text{diag}(1, \cos \gamma, \cos \gamma, 1). \quad (7)$$

III. TRANSPORT WITHOUT GATES

The most transparent solution to the kinetic equation is found by dividing all matrices in the problem into a diagonal part, denoted by the superscript d , and an off-diagonal part, denoted by the superscript od . In the case of the density matrix the diagonal part $f_{\mathbf{k}}^d$ represents the fraction of carriers which are in eigenstates of $H_{\mathbf{k}}$, while $f_{\mathbf{k}}^{od}$ is the fraction of carriers which are a continually changing mixture of eigenstates of $H_{\mathbf{k}}$. Diagonal matrices commute with $H_{\mathbf{k}}$ while off-diagonal matrices do not. The kinetic equation is correspondingly divided into equations for the diagonal and off-diagonal parts of the density matrix, which are coupled by scattering

$$\frac{\partial f_{\mathbf{k}}^d}{\partial t} + \hat{P}^d \hat{J}(f_{\mathbf{k}}) = \Sigma_{\mathbf{k}}^d \quad (8a)$$

$$\frac{\partial f_{\mathbf{k}}^{od}}{\partial t} + \frac{i}{\hbar} [H_{\mathbf{k}}, f_{\mathbf{k}}^{od}] = \Sigma_{\mathbf{k}}^{od} - \hat{P}^{od} \hat{J}(f_{\mathbf{k}}). \quad (8b)$$

\hat{P}^d and \hat{P}^{od} are projection operators which single out the diagonal and off-diagonal parts of matrices respectively. To solve these equations, we search for terms in the density matrix of lowest orders in $\hbar/(\varepsilon_F \tau)$ or equivalently those terms of lowest orders in n_i . Inspection of Eq. (8a) shows that, due to the absence of the commutator $[H_{\mathbf{k}}, f_{\mathbf{k}}^d]$, $f_{\mathbf{k}}^d$ starts at order n_i^{-1} while the leading term in $f_{\mathbf{k}}^{od}$ is independent of n_i (in other words order zero).⁴⁴ For weak momentum scattering therefore we only need to consider the effect of the scattering term acting on the diagonal part $f_{\mathbf{k}}^d$ of the density matrix. This reduces to the simple form

$$\hat{P}^d \hat{J}(f_{\mathbf{k}}^d) = \frac{n_i \lambda_k}{8\hbar^3 v^2} \int \frac{d\theta'}{2\pi} |\mathcal{U}_{\mathbf{k}\mathbf{k}'}|^2 F^d(\gamma) (f_{\mathbf{k}}^d - f_{\mathbf{k}'}^d), \quad (9)$$

with the diagonal matrix $F^d(\gamma)$ given by

$$F^d(\gamma) = \text{diag}(2, 1 + \cos 2\gamma, 1 + \cos 2\gamma, 2). \quad (10)$$

This matrix is due solely to the overlap of eigenstates at different wave vectors. The equation for the diagonal part $f_{\mathbf{k}}^d$ of the density matrix in the steady state, in which the time derivative can be dropped, reduces to

$$\frac{n_i \lambda_k}{8\hbar^3 v^2} \int \frac{d\theta'}{2\pi} |\mathcal{U}_{\mathbf{k}\mathbf{k}'}|^2 F^d(\gamma) (f_{\mathbf{k}}^d - f_{\mathbf{k}'}^d) = \Sigma_{\mathbf{k}}^d. \quad (11)$$

The solution to this equation is found simply as $f_{\mathbf{k}}^d = \Sigma_{\mathbf{k}}^d \tau_m$, where τ_m , which plays the role of a momentum scattering time, is a diagonal matrix given by

$$\tau_m^{-1} = \frac{n_i \lambda_k}{8\hbar^3 v^2} \int \frac{d\theta'}{2\pi} |\mathcal{U}_{\mathbf{k}\mathbf{k}'}|^2 F^d(\gamma) (1 - \cos \gamma) \quad (12)$$

with the matrix elements $\tau_{m11} = \tau_{m44} = \tau_+$ and $\tau_{m22} = \tau_{m33} = \tau_-$. This allows to write the electric-field-induced correction to the diagonal part of the density matrix as

$$f_{\mathbf{k}}^d = -\frac{2e\hbar v^2 \mathbf{E} \cdot \mathbf{k}}{\lambda_k} \text{diag}(\tau_+ \delta_1, \tau_- \delta_2, -\tau_- \delta_3, -\tau_+ \delta_4), \quad (13)$$

where we have used the abbreviation $\delta_s \equiv \delta(\varepsilon_{ks} - \varepsilon_F)$. We would like to stress that, although τ_m has the units of time, its elements do not correspond to actual scattering times and their energy-dependence does not come explicitly through any of the band energies, but rather through λ_k , which represents the difference between band energies. The form of λ_k implies that the dominant contribution to τ_m comes from t_\perp . Therefore, whereas $f_{\mathbf{k}}^d \propto n_i^{-1}$ does indicate that scattering tends to keep the Fermi surface near equilibrium, an expression for this function in terms of a momentum relaxation time cannot be formulated in bilayers.

The next step is to evaluate the off-diagonal part $f_{\mathbf{k}}^{od}$. Firstly, the effective source term in Eq. (8b) contains the off-diagonal projection of the scattering term acting on $f_{\mathbf{k}}^d$, given by $\hat{P}^{od} \hat{J}(f_{\mathbf{k}}^d)$. We find that this projection produces corrections of order $(\hbar v k / t_\perp)^2$ and therefore can be omitted. Secondly, when carrying out the time integral that determines the correction due to $\Sigma_{\mathbf{k}}^{od}$ we allow the field to have a time dependence $e^{-i\omega t}$, taking the limit $\omega \rightarrow 0$ at the end. (An unphysical negative conductivity may be obtained if this procedure is not followed.⁴⁴) $f_{\mathbf{k}}^{od}$ is found using the time evolution operator $e^{-iHt'/\hbar}$

$$f_{\mathbf{k}}^{od} = \lim_{\eta \rightarrow 0} \int_0^\infty dt' e^{-\eta t'} e^{i\omega(t'-t)} e^{-iH_{\mathbf{k}} t' / \hbar} \Sigma_{\mathbf{k}}^{od} e^{iH_{\mathbf{k}} t' / \hbar}, \quad (14)$$

with $\eta > 0$ a regularization factor. The time integral results in a series of δ -functions of the form $\delta(\varepsilon_{ks} - \varepsilon_{ks'})$, with $s \neq s'$. The only δ -function that can take nonzero values is $\delta(\varepsilon_{k2} - \varepsilon_{k3})$, due to the two bands that touch at $k = 0$. The problem of finding $f_{\mathbf{k}}^{od}$ therefore reduces to finding its matrix elements in the subspace spanned by $|u_{k2}\rangle$ and $|u_{k3}\rangle$. The only nonzero matrix element is

$$f_{\mathbf{k}23}^{od} = -\frac{i\pi e \mathbf{E} \cdot \hat{\boldsymbol{\theta}}}{2k} \lim_{\omega \rightarrow 0} [f_0(\varepsilon_{k2}) - f_0(\varepsilon_{k2} - \hbar\omega)] \delta\left(\varepsilon_{k2} - \frac{\hbar\omega}{2}\right) \quad (15)$$

Thus the bands that touch at $k = 0$ give $f_{\mathbf{k}}^{od}$, much like in single-layer graphene. Finally, the diagonal projection $\hat{P}^d \hat{J}(f_{\mathbf{k}}^{od})$, which would act as an effective source in the equation for $f_{\mathbf{k}}^d$, is also of order $(\hbar v k / t_\perp)^2$ and is omitted.

We determine separately the contributions to the electrical conductivity due to each term in the density matrix for electron doping, for which only the band labeled 2 is occupied. We require the expectation value of the current operator $\hat{\mathbf{j}} = -e\hat{\mathbf{v}}$, where $\hat{\mathbf{v}} = (1/\hbar)D\mathbf{H}_{\mathbf{k}}/D\mathbf{k}$ is the velocity operator in the basis of eigenstates of $H_{\mathbf{k}}$, and the covariant derivative $D/D\mathbf{k}$ has been defined above. $f_{\mathbf{k}}^d$, the fraction of carriers in eigenstates of $H_{\mathbf{k}}$, yields

(per valley and spin)

$$\sigma_d^{xx} = \frac{e^2 \zeta}{h} \frac{n\pi \hbar^2 v^2 t_\perp^2}{(4\pi^2 \hbar^4 v^4) n^2 + (4\pi \hbar^2 v^2 t_\perp^2) n + t_\perp^4} \quad (16)$$

with the dimensionless quantity $\zeta = \tau_- \lambda_k / \hbar$. In bilayers the screening wave vector is independent of the number density,³⁴ and $\zeta \propto n$ for long-range scatterers and is a constant for short-range scatterers. For short-range impurities, at low densities $\sigma_d^{xx} \propto n$ (Ref. 18) as in single-layer graphene;¹ but as the density increases nonlinear terms in n become more pronounced. For long-range impurities $\sigma_d^{xx} \propto n^2$ at low densities. The dependence of σ_d^{xx} on t_\perp indicates that σ_d^{xx} is due to carriers from both layers. Nevertheless, the leading term $\propto t_\perp^{-2}$, implying that at low densities interlayer tunneling hinders the transport of charge.

For a nonzero chemical potential μ , $f_{\mathbf{k}}^{od}$ gives (per valley and spin)

$$\sigma_{od}^{xx} = \frac{\pi e^2}{4h} \lim_{\omega \rightarrow 0} \left[\frac{1}{1 + e^{-\beta(\mu/2 + \hbar\omega/4)}} - \frac{1}{1 + e^{-\beta(\mu/2 - \hbar\omega/4)}} \right], \quad (17)$$

where $\beta = 1/kT$. To obtain the dc result at $T = 0$, one must take the limits $T \rightarrow 0$ and $\omega \rightarrow 0$; yet the result depends on the order in which these limits are taken. If $T \rightarrow 0$ first the result is $\pi e^2/(4h)$, whereas if $\omega \rightarrow 0$ first the result is zero. The same conundrum is present in single layers of graphene.⁴⁴ At present it is not clear whether this term is finite, and the theory presented in this paper up to now does not offer an indisputable solution. So far neither theory nor experiment can disambiguate this issue. Experiment could provide a conclusive answer if clean samples with zero carrier density were available, but that remains a daunting task. For weak momentum scattering, σ_{od}^{xx} is considerably smaller than σ_d^{xx} and cannot be extrapolated conclusively from a plot of n/n_i . Fortunately however, the analysis presented in the following section on biased bilayer graphene shows that, if we consider the conductivity due to a gate the answer can be found.

IV. GATE EFFECT ON TRANSPORT

We have determined so far the steady-state density matrix in unbiased graphene bilayers. We study next the interesting case of biased bilayer systems, in which the gap can be modified by the application of an external gate voltage V_g . The gate voltage gives rise to an additional term $H_{\mathbf{k}}^g$ in the Hamiltonian, which in the basis of eigenstates of $H_{\mathbf{k}}$ has the form

$$H_{\mathbf{k}}^g = -\frac{eV_g}{2\lambda_k} \begin{pmatrix} 0 & -\hbar v k & 0 & t_\perp \\ -\hbar v k & 0 & -t_\perp & 0 \\ 0 & -t_\perp & 0 & -\hbar v k \\ t_\perp & 0 & -\hbar v k & 0 \end{pmatrix} \quad (18)$$

We treat the gate potential $-eV_g$ in first-order perturbation theory. It is easily checked that to first order in

$|eV_g|$ and $\hbar v k/t_\perp$ the contribution of $H_{\mathbf{k}}^g$ to the scattering term $\hat{J}(f_{\mathbf{k}})$ in the kinetic equation is zero. The gate gives rise to an additional source term in the kinetic equation, which takes the form

$$\Sigma_{\mathbf{k}}^g = \frac{e\mathbf{E}}{2\hbar} \cdot \frac{D}{D\mathbf{k}} \left\{ H_{\mathbf{k}}^g, \left(\frac{\partial f_0}{\partial \mathbf{k}} \right) \left(\frac{\partial H}{\partial \mathbf{k}} \right)^{-1} \right\} - \frac{i}{\hbar} [H_{\mathbf{k}}^g, f_{\mathbf{k}}]. \quad (19)$$

The bracket $\{\cdot\}$ denotes the symmetrized dot product $\{\mathbf{a} \cdot \mathbf{b}\} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. Once again, the covariant derivative $D/D\mathbf{k}$ appears instead of the ordinary derivative. We proceed exactly as before in order to find the additional correction to the density matrix due to the gate, which we call $f_{\mathbf{k}}^g$. The nonzero diagonal terms in $f_{\mathbf{k}}^g$ are

$$f_{\mathbf{k}22}^g = -f_{\mathbf{k}33}^g = -\frac{\pi e^2 V_g t_\perp \tau_- \mathbf{E} \cdot \hat{\boldsymbol{\theta}}}{4\hbar k \lambda_k} \lim_{\omega \rightarrow 0} [f_0\left(\frac{\hbar\omega}{2}\right) - f_0\left(\frac{-\hbar\omega}{2}\right)] \delta\left(\varepsilon_{k2} - \frac{\hbar\omega}{2}\right). \quad (20)$$

Several other terms either vanish as $\omega \rightarrow 0$ or give zero contributions to steady-state expectation values after integration over wave vector. These terms are omitted here for simplicity and without loss of generality.

Following the same procedure as in the unbiased case discussed above, we find that the nonequilibrium correction $f_{\mathbf{k}}^g$ to the density matrix due to the gate gives only an *off-diagonal* conductivity (per valley and spin)

$$\sigma_g^{xy} = -\frac{\pi^2 e^3 V_g \tau_-}{2\hbar^2} \lim_{\omega \rightarrow 0} \left[\frac{1}{1 + e^{-\beta(\mu/2 + \hbar\omega/4)}} - \frac{1}{1 + e^{-\beta(\mu/2 - \hbar\omega/4)}} \right]. \quad (21)$$

The term σ_g^{xy} , like σ_{od}^{xx} and like the term analogous to σ_{od}^{xx} in single-layer graphene,⁴⁴ depends on the order of limits. Nevertheless, this term is the *only* contribution to the off-diagonal conductivity, a fact that can clarify the correct order of limits. Crystal symmetry and Onsager relations imply that an off-diagonal conductivity σ_{xy} requires time reversal symmetry breaking, for example through the presence of a magnetic field. Therefore it must be argued on physical grounds that an off-diagonal term in the conductivity such as that found in the current work should not exist.⁴⁶ The only resolution to this is to take the limit $\omega \rightarrow 0$ first. These observations imply that σ_{od}^{xx} in bilayer and single-layer graphene should be

zero.

We note that the findings of this section do not conflict with the presence of a minimum conductivity $\sigma_{xx} = 4e^2/(\pi h)$ in ballistic graphene, which has been measured experimentally.⁴⁵ The ballistic regime is qualitatively different in that the mean free path greatly exceeds the sample size.

We thank E. McCann, A. H. MacDonald, Y. Barlas, S. Adam, E. Rossi and E. H. Hwang for enlightening discussions. The research at Argonne National Laboratory was supported by the US Department of Energy, Office of Science, Office of Basic Energy Sciences, under Contract No. DE-AC02-06CH11357.

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